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A singularly perturbed parabolic problem that degenerates on one boundary†

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ABSTRACT

The singularly perturbed parabolic problem $\varepsilon u_{xx}(x,t) - x^\alpha u_t(x,t) = f(x,t)$ is considered on the rectangular domain $G := (0,1)\times(0,T]$ with Dirichlet initial and boundary data. Here $\varepsilon > 0$ is a small parameter and $\alpha > 0$ is a fixed constant. The differential operator degenerates at $x = 0$, so standard techniques for singularly perturbed equations cannot be used here. This problem was studied in [1] under the assumption that $f(x,t) = x^\alpha f_1(x,t) + \varepsilon^{\alpha/(2+\alpha)} f_2(x,t)$, which implies that the solution $u(x,t)$ is uniformly bounded (in the maximum norm) with respect to ε; the tensor product of a spatial Shishkin-type mesh with three transition points was taken with a piecewise equidistant temporal mesh, then it was proved in [1] that for $\alpha \geq 1$ a standard finite difference scheme was $O(N_x^{-\beta} + N_x^{-1}\ln N_x + N_t^{-1})$ convergent in the discrete maximum norm, uniformly in ε, where $\beta = 4/(2+\alpha)$ and N_x and N_t are the numbers of mesh points used in space and time respectively.

In the present work the more general case $f(x,t) = x^{\gamma_1} f_1(x,t) + \varepsilon^{\gamma_2} f_2(x,t)$ is considered, where γ_1 and γ_2 are fixed non-negative numbers. The solution u is not bounded uniformly in ε if $\gamma_1 < \alpha$ or if $\gamma_2 < \alpha/(2+\alpha)$. The transition points in the mesh of [1] are modified and it is shown that for the same finite difference scheme the order of convergence then improves significantly. The analysis is complicated and includes both the cases when u is and is not uniformly bounded in ε. Moreover, the special derivatives u_{xxx} and u_{xxxx} blows up as $x \to 0$ for $0 < \alpha < 1$. Thus usual techniques to bound the consistency error can not be applied. A special barrier function is constructed to prove the convergence of the scheme also for $0 < \alpha < 1$. This approach can be easily modified for another problems with unbounded spatial derivatives. Numerical examples supporting the theoretical results are provided.

References

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