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Optimal Error Estimate of Upwind Scheme on Shishkin-type meshes for Singularly Perturbed Parabolic Problems with Discontinuous Convection Coefficients

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ABSTRACT

Here, we consider the following class of singularly perturbed parabolic convection-diffusion problems with discontinuous convection coefficients posed on the domain $G^- \cup G^+,\ G^- = (0,\xi) \times (0,T],\ G^+ = (\xi,1) \times (0,T]$:

$$\left(\varepsilon \frac{\partial^2 u}{\partial x^2} + a(x)\frac{\partial u}{\partial x} - b(x)u - \frac{\partial u}{\partial t}\right)(x,t) \;=\; f(x,t), \quad (x,t) \in G^- \cup G^+,$$

where $0 < \varepsilon \ll 1$ is a small parameter, the coefficients $a,\ b$ and the source term f are sufficiently smooth functions on $G^- \cup G^+$ such that $a(x) < 0,\ x < \xi, \quad a(x) > 0,\ x > \xi, \quad b(x) \geq 0 \quad \text{on } \overline{\Omega} = [0,1]$, with suitable initial, boundary and the interface conditions at $x = \xi$. In general, the solutions of this class of problems possess strong interior layers which are basically thin regions in the neighbourhood of the interior of the domain where the gradients of the solutions steepen as the perturbation parameter ε tends to zero.

To solve these problems, we apply the implicit upwind finite difference scheme which comprises of the classical backward-Euler method for the time discretization and the simple upwind scheme for the spatial discretization. While at the point of discontinuity, a first-order one-sided difference approximations are used to keep the continuity of the spatial derivative. The finite difference scheme is analyzed on Shishkin-type meshes which include the classical piecewise-uniform Shishkin mesh and the Bakhvalov-Shishkin mesh. These meshes are condensed around the interior layers and constructed with the help of appropriate mesh generating functions which basically provides a theoretical framework for studying the convergence analysis of the implicit upwind scheme in a unified way.

We derive suitable conditions on the mesh-generating functions that are sufficient for the convergence of the method, uniformly with respect to the perturbation parameter ε. Utilizing these conditions, we show that the implicit upwind scheme converges uniformly in the discrete supremum norm with an optimal error bound of $\mathrm{O}\big(N^{-1} + \Delta t\big)$ on the Bakhvalov-Shishkin mesh, while on the piecewise-uniform Shishkin mesh it is of $\mathrm{O}\big(N^{-1} \ln N + \Delta t\big)$. Here, N is the number of mesh-intervals in the spatial direction and Δt is the step size in the temporal direction. Finally, numerical results are presented to validate the theoretical results.

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